Introduction

[Updated: 01/02/2018]

Agenda:

- Origins of Contemporary Philosophy of Language: Reference, Meaning, Names, Descriptions (Frege, Russell et al.)
- Proper Names, Direct Reference, Semantic Externalism (Russell, Searle, Kripke, Putnam, Evans)
- Propositional Attitude Reports (Frege, Quine, Kaplan, Kripke et al.)

The Problem of Linguistic Meaning:

- Certain sorts of marks on paper and noises we make when we speak have *meanings*.
- We *grasp* those meanings *without* even thinking about it.
- This fact is *very* striking!
- *How* do we manage to do it?
- *What* is it in virtue of which the string has the distinctive meaning it does?
    - We understand individual English words and linguistic expressions and we understand something about how they are strung together.
    - Simple Referential Theory: Linguistic expressions have the meanings they do because they *stand for* things; what they mean is what they stand for. Words are like *labels*. They are conventionally/arbitrarily associated with things.

- ‘Yuri Balashov’ denotes Yuri Balashov.
- ‘dog’, ‘chien’ and ‘Hund’ all denote dogs (in English, French and German).
- ‘The cat is on the mat’ *represents* some cat’s being on the mat, presumably in virtue of ‘The cat’ *designating* the cat, ‘the mat’ *designating* the mat, and ‘is on’ *denoting* the relation of being-on.
- Sentences *mirror* states of affairs they describe.
- The meaning of a linguistic term is just the thing/entity it designates, and the meaning of a sentence is a matter of understanding what it says based on one’s knowledge about what the components of the sentence refer to.

- Problem: Not every term denotes anything.
    - Non-existent things: ‘Sherlock Holmes’, ‘Santa Claus’, ‘unicorn’
    - ‘There is a dearth of champagne in the ice-box.’
    - ‘I saw nobody’.
    - ‘The professor is tall’. (What does ‘is’ refer to?)
    - ‘sake’, ‘behalf’, ‘dint’
    - ‘very’, ‘of’, ‘the’, ‘yes’
- There is *more* to meaning than just naming something…
- Narrow down the problem: focus on Singular Terms – expressions purporting to denote or designate particular individual objects: people, places, times, etc., as opposed to general terms that can apply to more than one thing (‘dog’, ‘fat’).

  - Proper names: ‘Ted Turner’, ‘Trafalgar Square’, ‘1:55 pm, Jan 5, 2018’, ‘12365’
  - Definite descriptions: ‘the current U.S. president’, ‘Dr. Jagnow’s office’, ‘Chuck Cross’s favorite logic book’
  - Personal pronouns: ‘I’, ‘she’
  - Demonstratives: ‘this’, ‘that’

- Perhaps Simple Referential Theory is true at least of singular terms?

- Reference to nonexistents:
  - ‘The present King of France is bald’.

- Negative existentials:
  - ‘The present King of France does not exist’.
  - ‘Superman (Clark Kent) does not exist’.

  - ➢ Seem to be true and seem to be about the present King of France and about Clark Kent. But if the first sentence is true, it can’t be about the present King of France, because there is no such King for it to be about. On the other hand, if it is about the present King of France, then it is false, for the King must then in some sense exist.

- Frege’s problem about identity statements:
  - The Morning Star = The Evening Star
  - The Morning Star = The Morning Star

- Substitutivity of co-referential expressions:
  - Cicero = Tully (Mark Tully Cicero: 106 BC – 43 BC)
  - John believes that Cicero denounced Catiline.
  - John believes that Tully denounced Catiline.

I. Machinery: Set Theory and Set-Theoretic Constructions

Excellent reasons to learn / review some Set Theory in this course

- The emergence and development of modern Set Theory is bound up with the origins of modern Logic, Foundations of Mathematics and Philosophy of Language (from 1870s on).
- Starting with a very simple vocabulary the language of Set Theory quickly and systematically generates (with the aid of just Predicate Logic) an enormously rich structure of *abstract objects* sufficient to formalize all mathematical and related disciplines. Sets and Set Theory are important foundational tools in contemporary Mathematics, Linguistics, and Analytic Philosophy.
- In particular, they allow us to regiment many semantical concepts (e.g. interpretations and models) required for understanding some portions of Philosophy of Language.
- Set-theoretic paradoxes underlie truth-theoretic and other semantical paradoxes.
- A true and rigorous science of the Infinite.
- A good exercise in abstraction and brain gymnastics.

Good sources (I've made good use of both):

- SEP: plato.stanford.edu
- Discrete Structures / Discrete Mathematics, by Shunichi Toida: http://www.cs.odu.edu/~toida/nerzic/content/web_course.html

Elements of Basic / Naïve Set Theory

1. What are Sets?

- ✓ Informally: *collections* of *entities* of *any sort* – natural or unnatural, homogenous or inhomogeneous, haphazard or systematic.
- ✓ The entities "in" the collection are the *members* of a set.
- ✓ Anything can be "in" a set. There are sets of numbers, people, other sets.

2. Notation

{2, 5}

The set containing 2 and 5

{YB, {Marseille}, 19}

The set containing the following three members:

- me
- the set containing Marseille (= the *unit*/*singleton set* of Marseille)
- the number 19

{0, 1, 2, ...}

The infinite set containing all the natural numbers.

{*x*: *x* is a prime number}

The (infinite) set containing the prime numbers, i.e. {2, 3, 5, 7 ...}

{*x*: *x* is purple}

The set of all purple things

$\{x: \phi(x)\}$
The set of all things having the property denoted by ‘ϕ’

$x \in y$: x is a member of y
x belongs to y
x is an element of y

$x \notin y$: x is not a member of y
Etc.

$2 \in \{2, 5\}$

$3 \notin \{2, 5\}$

{Marseille}∈{YB, {Marseille}, 19}

Marseille ∉{YB, {Marseille}, 19}

{YB}∉{YB, {Marseille}, 19}

- ✓ Set Theory = A theory of the relation ∈

3. Important Notions, Definitions, Operations and Principles

3.1. *Extensionality*:

Two sets are *identical* (i.e. are *the very same set*) iff they have exactly the same members.
$x=y$ iff $\forall z\, (z \in x \leftrightarrow z \in y)$

- ✓ Sets are *fully/exhaustively defined* by their members.

$\{2, 5\} = \{5, 2\} = \{2, 5, 2\}$

{YB, {Marseille}} ≠ {YB, Marseille}

3.2. *Subset* (⊆):

x is a *subset* of y iff all the members of x are members of y.
$x \subseteq y$ iff $\forall z\,(z \in x \rightarrow z \in y)$

$\{2, 5\} \subseteq \{2, 3, 4, 5, 7\}$

$\{x$: x is a prime number$\} \subseteq \{0, 1, 2, ...\}$

✓ Any set is a subset of itself: $x \subseteq x$
E.g. $\{2, 5\} \subseteq \{2, 5\}$

3.4. *Proper Subset* (⊂):

x is a *proper subset* of y iff (i) x is a subset of y and (ii) $x \neq y$.
$x \subset y$ iff $(x \subseteq y \wedge x \neq y)$

$\{2, 5\} \subset \{2, 3, 4, 5, 7\}$
$\{2, 5\} \not\subset \{2, 5\}$

3.5. *Empty Set (Null Set)* (∅):

A set that has no elements is called an *empty set* (∅).
$\forall x\,(x \notin \varnothing)$

✓ *There is* such a set (i.e. an empty set).
✓ There is *only one* such set (by Extensionality).

✓ $\varnothing$ *versus* $\{\varnothing\}$

✓ The empty set is a subset of *any* set:
$\forall x\,(x$ is a set $\rightarrow \varnothing \subseteq x)$

3.6. *Power Set* ($\wp$):

The *power set* of *x* ($\wp(x)$) = the set of all subsets of *x* (including *the empty set* and *x itself*):

$\forall z\, (z\in \wp(x) \leftrightarrow \forall y\, (y\in z \rightarrow y\in x) \leftrightarrow z\subseteq x)$

$\wp\{2, 5\} = \{\varnothing, \{2\}, \{5\}, \{2, 5\}\}$

✓ If *x* has *n* members $\wp(x)$ has 2^n members.

3.7. *Union* ($\bigcup$):

The *union* of a *set of sets* *x* ($\bigcup x$) is a set *y* which has as members all of the members of all of the members of *x*:

$y = \bigcup x$ iff $\forall z\, [z\in y \leftrightarrow \exists w\, (w\in x \wedge z\in w)]$

$\bigcup \{\{2, 5\}, \{1, 3, 5\}, \{7, 9\}\} = \{1, 2, 3, 5, 7, 9\}$

The union of *two* sets *x* and *y* ($x \bigcup y$):

$x \bigcup y = \{z: (z\in x \vee z\in y)\}$

3.8. *Intersection* ($\bigcap$):

The *intersection* of a *set of sets* *x* ($\bigcap x$) is a set *y* consisting of all and only the *common members* of all the sets in *x*:

$y = \bigcap x$ iff $\forall z\, [z\in y \leftrightarrow \forall w\, (w\in x \rightarrow z\in w)]$

$\bigcap \{\{2, 5\}, \{1, 3, 5\}, \{5, 9\}\} = \{5\}$

The intersection of *two* sets *x* and *y* ($x \bigcap y$):

$x \bigcap y = \{z: (z\in x \wedge z\in y)\}$

4. Ordered Pairs, Relations, Cartesian Products, Functions

4.1. *Ordered Pair* ($<x,y>$):

An *ordered pair* ($<x,y>$) is a pair of objects with an *order* imposed on them.

Two ordered pairs $<x,y>$ and $<z,w>$ are *identical* iff $x=z$ and $y=w$: $<x,y> = <z,w>$ iff $x=z \wedge y=w$.

Set-theoretic definition of *ordered pair* (Kuratowski 1921):
$<x,y> =_{df} \{\{x\},\{x,y\}\}$

Ordered triple: $<x,y,z> =_{df} <<x,y>,z>$

Ordered quadruple: $<x,y,z,w> =_{df} <<x,y,z>,w>$ Etc.

Alternative definitions (arguably inferior to Kuratowski's):
Norbert Wiener (1914): $<x,y> =_{df} \{\{\{x\}, \emptyset\}, \{\{y\}\}\}$
Felix Hausdorff (1914): $<x,y> =_{df} \{\{x, a\}, \{y, b\}\}$, where a and b are two distinct objects different from x, and y.

4.2. *Cartesian Product* ($X\times Y$)

The set of all ordered pairs $<x,y>$, where $x\in X$ and $y\in Y$, is called the Cartesian Product of X and Y:
$X\times Y =_{df} \{<x,y>: (x\in X \wedge y\in Y)\}$

4.3. *Relations*

Familiar animals:

x is a father of y

$x = y^2$

$x>y$

Set theoretically, a *binary relation R* from a set *X* to a set *Y* is a *set of ordered pairs* $<x,y>$ where $x \in X$ and $y \in Y$.

Notation: xRy (for $<x,y> \in R$)

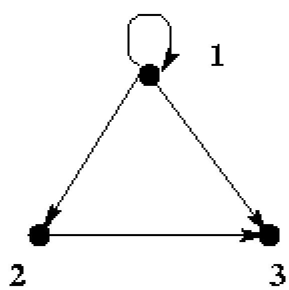
Domain of *R* (dom (*R*)) is the set of all *x* standing in *R* to some *y*:
dom (*R*) = $\{x: \exists y\ xRy\}$.

Range of *R* (ran (*R*)) is the set of all *y* to which some *x* (one or more) stand in *R*:
ran (*R*) = $\{y: \exists x\ xRy\}$.

- ✓ A binary relation *R* from *X* to *Y* is a *subset* of Cartesian product $X \times Y$.

4.4. *Digraphs*

Digraph (= *directed graph*) – a diagram composed of *vertices* (nodes) and *arcs* (arrows) connecting vertices.



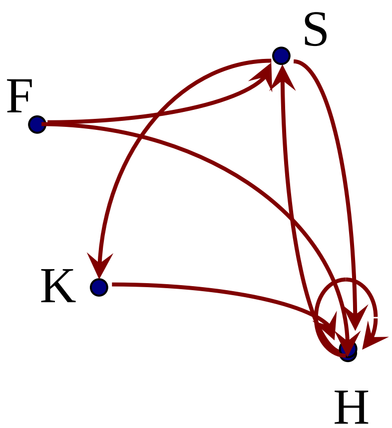
[Figure source:
http://www.cs.odu.edu/~toida/nerzic/content/digraph/definition.html]

- ✓ Digraphs provide a convenient way to illustrate/model binary relations, where the *relata* (objects standing in binary relations) are represented by vertices and the *relations* (i.e. ordered pairs) by arrows.

More formally (see http://www.cs.odu.edu/~toida/nerzic/content/digraph/definition.html for details):

> A *digraph G* can be said to be an *ordered pair of sets G* = <*V*, *A*>, where *V* is a set of *vertices* and *A* is a set of ordered pairs (*arcs*) of vertices from *V*.

E.g., this digraph (G_{Phil})



is simply <{F, H, S, K}, {<F,H>, <H,S>, <S,K>, <F,S>, <H,H>, <K,H>, <S,H>}>.

- ✓ Viewed this way, G_{Phil} is a pretty abstract object – eventually, a set… (No surprise: *everything* is a set...)

4.5. *Properties of Binary (Dyadic) Relations*

A relation R on a set X is:

Reflexive iff $\forall x \in X$: xRx

- E.g. (formal): $\{<a,a>, <a,c>, <b,b>, <c,c>, \}$ on the set $\{a, b, c\}$
- E.g. (informal): *having the same length as* on a set of extended objects; $\leq$ on a set of numbers (natural, rational or real)

Symmetric iff $\forall x,y \in X$: $xRy \rightarrow yRx$

- E.g. (formal): $\{<a,b>, <b,a>, <a,a>, <c,c>, \}$ on the set $\{a, b, c\}$
- E.g. (informal): *being a sibling of* on a set of people; $=$ on any set of numbers

Transitive iff $\forall x,y,z \in X$: $(xRy \wedge yRz) \rightarrow xRz$

- $\{<a,c>, <a,b>, <c,b>, <c,c>, \}$ on the set $\{a, b, c\}$
- E.g. (informal): *being heavier than* on a set of physical objects on Earth; $<$, $>$, $\leq$, $\geq$, $=$ on a set of numbers; *being a part of* on any set of entities composed of parts

Nonreflexive iff R is not reflexive

Nonsymmetric iff R is not symmetric

Nontransitive iff R is not transitive

Irreflexive iff $\forall x \in X$: $\sim xRx$

Asymmetric iff $\forall x,y \in X$: $xRy \rightarrow \sim yRx$

Intransitive iff $\forall x,y,z \in X$: $(xRy \wedge yRz) \rightarrow \sim xRz$

Antisymmetric iff $\forall x,y \in X$: $(xRy \wedge yRx) \rightarrow x = y$
$\forall x,y \in X$: $(xRy \wedge x \neq y) \rightarrow \sim yRx$

✓ *Asymmetry* → *Nonsymmetry*
But ~ (*Nonsymmetry* → *Asymmetry*)

✓ *Intransitivity* → *Nontransitivity*
But ~ (*Nontransitivity* → *Intransitivity*)

✓ *Irreflexivity* → *Nonreflexivity*
But ~ (*Nonreflexivity* → *Irreflexivity*)

[Some examples here]

Equivalence Relation:

A binary relation R on a set X is an *equivalence relation* iff it is

(i) symmetric;
(ii) transitive;
(iii) reflexive.

E.g.: *being the same age as, being a compatriot of*

✓ An equivalence relation R *partitions* set X (of objects on which it is imposed) into non-overlapping subsets.

Partial Order:

A binary relation R on a set X is a *partial order* iff it is

(i) reflexive;
(ii) antisymmetric;
(iii) transitive.

✓ When R is a partial order on a set X, the ordered pair $\langle X, R\rangle$ is called a *poset* (*partially ordered set*)

Total Order:

A binary relation R on a set X is a *total* (*linear*) *order* iff it is

(i) a partial order;
(ii) $\forall x,y \in X: \quad xRy \vee yRx$.

4.6. *Functions*

- ✓ Set-theoretically, a *function F* is a *special type of relation* where every object *x* from a domain is related to *one and only one* object $F(x)$ called the *value* of the function at *x*.

Function:

A *function* (*mapping, correspondence*) *F from* a set *X to* (*into*) a set *Y* ($F: X \mapsto Y$) is a relation from *X* to *Y*, such that:

(i) $\forall x \in X \quad \exists y \, (= F(x)) \in Y \qquad <x, y> \in F$;

(ii) $\forall x_1, x_2 \in X, \quad (x_1 = x_2) \rightarrow [F(x_1) = F(x_2)]$

X: the *domain* of *F*

$\{F(x): x \in X\} \subseteq Y$: the *range* of *F*

One-to-one (*Injection*):

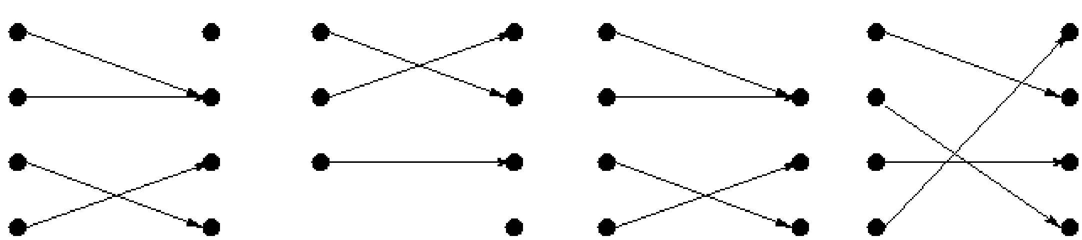
F is *one-to-one* (*injective*) iff $[F(x_1) = F(x_2)] \rightarrow (x_1 = x_2)$

Onto (*Surjection*):

F is *onto* (*surjective*) iff $\forall y \in Y \quad \exists x \, [F(x) = Y]$

Bijection:

F is *a bijection* iff *F* is both *onto* and *one-to-one*.



A function | **A one-to-one function (Not onto)** | **An onto function (Not one-to-one)** | **A bijection**

[Figure source: http://www.cs.odu.edu/~toida/nerzic/content/function/definitions.html]

5. Numbers, Infinity, Cardinality

5.1. *Natural Numbers*:

Start with $\varnothing$.

Define the number 0 as $\varnothing$: $0 =_{df} \varnothing$

Define the number 1 as $\{\varnothing\}$: $1 =_{df} \{\varnothing\}$

Next define the number 2 as: $2 =_{df} \{0,1\} = \{\varnothing, \{\varnothing\}\}$

" – " 3 as: $3 =_{df} \{0,1,2\} = 2 \cup \{2\} = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$

" – " 4 as: $4 =_{df} \{0,1,2,3\} = 3 \cup \{3\} =$
$\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}$

…

" – " n+1 as: $n+1 =_{df} \{0,1,2, \ldots n\} = n \cup \{n\}$

$S(n) =_{df} n \cup \{n\}$: the *successor* of the set (= number) n

Inductive Set:

A set I is called *inductive* if:

(i) $0 \in I$;
(ii) if $n \in I$ then $n+1 \in I$.

The Set of All Natural Numbers ($\mathbb{N}$):

The Set of All Natural Numbers ($\mathbb{N}$) is the (smallest) inductive set I.

✓ $\mathbb{N}$ is *infinite*. What does this mean?

5.2. *Cardinality* ($|\ |$):

Sets X and Y have the *same cardinality* if there is a one-to-one function F with domain X and range Y:
$|X| = |Y|$

$|X| \leq |Y|$ iff there is a one-to-one function (mapping) from X into (but not *onto*) Y: $X \mapsto Y$.

$|X| < |Y|$ iff $|X| \leq |Y|$ and $|X| \neq |Y|$

✓ $|X| < |Y|$ means that there is a one-to-one mapping of X onto a *subset* of Y but no one-to-one mapping of X onto Y.

Finite Set:

A set X is *finite* iff X has the same cardinality as some natural number $n \in \mathbb{N}$.

✓ If so then we can define $|X| = n$ and say that X has n elements.

Infinite Set:

A set X is *infinite* iff it is not finite.

Countable Set:

A set X is *countable* iff $|X| = |\mathbb{N}|$.

✓ X is countable if there is a one-to-one mapping from $\mathbb{N}$ *onto* X, i.e. if X is in the *range* of an *infinite one-to-one sequence.*

- ✓ Cardinality is a generalization of the (intuitive) notion of "size" to infinite sets.

From now on we will be somewhat informal…

- ✓ Set *X* can (usefully) be said to be *smaller than* set *Y* (i.e. $|X| < |Y|$) iff there is a one-to-one mapping relating each member of *X* to a distinct member of *Y* but there is no one-to-one mapping relating each member of *Y* to a distinct member of *X*.

- ✓ $|X| = |Y|$ (*X* and *Y* are "the same size") iff there is a one-to-one mapping relating each member of *X* to a distinct member of *Y* and there is a one-to-one mapping relating each member of *Y* to a distinct member of *X*.

- ✓ For finite sets, cardinality is the ordinary notion of the size of a set.

- ✓ An *infinite* set can have the *same* cardinality as one of its *proper* subsets.

- ✓ A set is infinite iff it can be put into one-to-one correspondence with a *proper subset* of itself.

✓ The following sets are *countable*, i.e. their members can be put into one-to-one correspondence with the members of the set of natural numbers ℕ.

1	2	3	4	5	6	7	…
↕	↕	↕	↕	↕	↕	↕	↕
2	4	6	8	10	12	14	…
↕	↕	↕	↕	↕	↕	↕	↕
½	¼	1/8	1/16	1/32	1/64	1/128	…
↕	↕	↕	↕	↕	↕	↕	↕
a_1	a_2	a_3	a_4	a_5	a_6	a_7	…
↕	↕	↕	↕	↕	↕	↕	↕

All rational numbers

- Theorem (Georg Cantor, 1874): The set of points $\mathbb{R}$ of a finite segment (e.g. [0;1]) of a real line (i.e. the *set of real numbers between 0 and 1*) is uncountable (i.e. not countable).

Proof. By *reductio*. Assume that $\mathbb{R}$ is countable. Then there exists a one-to-one mapping of $\mathbb{N}$, the set of natural numbers, to $\mathbb{R}$ (the set of real numbers x, such that $0 \leq x \leq 1$).

1	↔	.	$\boldsymbol{a^1_1}$	a^1_2	a^1_3	…	a^1_m	…
2	↔	.	a^2_1	$\boldsymbol{a^2_2}$	a^2_3	…	a^2_m	…
3	↔	.	a^3_1	a^3_2	$\boldsymbol{a^3_3}$	…	a^3_m	…
…	↔	.	…	…	…	…	…	…
n	↔	.	a^n_1	a^n_2	a^n_3	…	a^n_m	…
…	↔	.	…	…	…	…	…	…

We will show that there is at least one real number r, which is *not* included in this mapping. Let:

$r = \; . \; b_1 \; b_2 \; b_3 \; \ldots \; b_m \; \ldots$

where: $b_m = a^m_m + 1$ if $0 \leq a^m_m < 9$;
$b_m = 0$ if $a^m_m = 9$.

r is different from *any* real number included in the mapping. ■

Cardinality of $\mathbb{N}$: $|\mathbb{N}| = \aleph_0$
Cardinality of $\mathbb{R}$: $|\mathbb{R}| = c > \aleph_0$

$1, 2, \ldots \aleph_0, \ldots c, \ldots$

✓ The number of points in a line (i.e., the set of real numbers) is *uncountably infinite*.

II. Machinery: Syntax, Semantics, Pragmatics

- Syntax:
  - The study of *formation* and *derivation rules* of a formal or natural language (e.g., English, Chinese, C++, propositional or predicate logic, etc.)
  - How to construct *well-formed expressions* from smaller units of a language (*formation* / *concatenation rules*), e.g.:
    - English sentences from English words: 'The' + 'cat' + 'is' + 'on' + 'the' + 'mat' → 'The cat is on the mat'
    - Longer English sentences from shorter English sentences: 'Snow is white' + 'and' + 'Grass is green' → 'Snow is white and grass is green'.
    - Sentences of propositional (sentential) logic from atomic sentences and logical symbols: '(' + 'A' + '⊃' + 'B' + ')' + '&' + '(' + 'C' + '∨' + 'D' + ')' → '(A ⊃ B) & (C ∨ D)'
  - Given a sequence of well-formed expressions of a language, what *other* expressions one is entitled to write below them (*derivation rules*, mostly in formal languages), e.g.:

Algebra:

$a > b$
$b > c$
$\therefore\ a > c$

Propositional Logic (*Modus Tollens*):

A ⊃ B
A
∴ B

✓ Important: Syntax is a matter of (properly) manipulating *formal* symbols while *completely abstracting* from their *meaning*. In fact, the symbols may *lack* any meaning *whatsoever*, yet constructions from them may be syntactically correct (or incorrect).

Example: The Syntax of Sentential Logic (SL)

- What are the basic expressions (the vocabulary) of SL?
- How can we build other expressions from the basic ones?

Def: The *vocabulary* of SL consists of the following:

1. The *sentence letters* of SL (= the *atomic sentences* of SL): the capital Roman letters 'A' through 'Z' with and without positive integer subscripts.
2. The *sentential connectives* of SL: '~', '&', '∨', '⊃', '≡'.
3. The *punctuation marks* of SL are '(' and ')'.

Def: The *metavariables* of SL:

The boldfaced letters '**P**', '**Q**', '**R**', and '**S**' (with and without positive integer subscripts) are metavariables ranging over strings of sentence letters, connectives, and punctuation marks of SL. The Greek capital letters '**Γ**' and '**Δ**' (with and without positive integer subscripts) are metavariables ranging over sets of sentences of SL.

Def: The *complete set of sentences* of SL:

The smallest set $\mathcal{S}$ containing every sentence letter (i.e. atomic sentence) of SL and such that:

1. If $\mathbf{P} \in \mathcal{S}$ then $\sim\mathbf{P} \in \mathcal{S}$;
2. If $\mathbf{P}, \mathbf{Q} \in \mathcal{S}$ then $(\mathbf{P}\&\mathbf{Q}), (\mathbf{P}\vee\mathbf{Q}), (\mathbf{P}\supset\mathbf{Q}), (\mathbf{P}\equiv\mathbf{Q}) \in \mathcal{S}$.

Def: *immediate sentential component*; *sentential component*; *main connective* of a sentence of SL:

1. If **P** is an atomic sentence, it has no main connective, no immediate sentential components, and is its only sentential component.
2. If **P** is of the form **~Q** (a negation), the only immediate sentential component of **P** is **Q**, the main connective of **P** is the token of '~' preceding **Q** in **P**, and the sentential components of **P** are **P** itself, **Q**, and all sentential components of **Q**.
3. If **P** is of the form **(Q&R)** (a conjunction), **(Q∨R)** (a disjunction), **(Q⊃R)** (a material conditional), or **(Q≡R)** (a material biconditional), then the immediate sentential components of **P** are **Q** and **R**, the main connective of **P** is the connective token appearing between **Q** and **R** in **P**, and the sentential components of **P** are **P** itself, **Q** and **R**, and all sentential components of **Q** and of **R**.

Def: An *atomic component* of a sentence of SL is its sentential component that is an atomic sentence (a sentence letter) of SL.

Def: A *proper sentential component* of a sentence of SL is its sentential component that is distinct from the sentence itself.

- Semantics:
  - In contrast with syntax, semantics is all about the *meaning* of linguistic expressions – about *word-world relations*.
  - A semantic theory must explain how language (a formal or a natural language) "hooks" onto the world.
  - Central semantic notions: *truth* and *reference*. Informally:
  - *Truth* is the main semantic value of *declarative sentences*.
    - A sentence is true in virtue of the correspondence between what it says and the way the world is. E.g., the sentence 'Snow is white' is true iff snow is white.
  - *Reference* is the main semantic value of linguistic *terms* that are parts of sentences (primarily, of *singular terms*).
    - A singular term has reference in virtue of *denoting* an item outside language (i.e. in the world, broadly speaking). E.g., 'Snow' refers to snow (the white stuff); 'the current U.S. president' refers to the current U.S. president, etc.

  ✓ As you expect, there should be a systematic relation between the truth of sentences of a language and the reference of its terms. E.g., 'Snow is white' is true because 'snow' refers to a stuff that is white.

  ✓ The exact *nature* of the relationship between truth and reference is a big issue in the philosophy of language. Some philosophers (e.g. Frege) tried to assimilate truth to reference (we'll see how this works – or is supposed to work – shortly…).

- Pragmatics:
  - Is about the *use* of language by speakers in *context*.
  - Focuses on ways in which the context of use influences or even determines reference and other semantical values of expressions and sentences.

✓ (Another BIG issue, as you may guess…)

[The subsequent sections of these Introductory Notes are largely based on Martinich: pp. 1–26.]

III. Machinery: Use and Mention, etc.

- We *use* words and other linguistic expressions to talk about things *other than* words and expressions themselves. Words are signs that point *beyond* themselves to other things.

E.g., we use ‘Minnesota’ to talk about a political subdivision of the United States:

Minnesota was the thirty-second state admitted to the Union. (1)

- We *mention* words when we use them to talk about *themselves*. E.g.:

‘Minnesota’ is an Indian word. (2)

Compare:

‘Minnesota’ was the thirty-second state admitted to the Union. (3)

Minnesota is an Indian word. (4)

(1) and (2) are true; (3) and (4) are false.

In sentence (1), ‘Minnesota’ is used to talk about Minnesota. (5)

- Is ‘Minnesota’ used or mentioned in (5)?

In sentence (2), ‘ ‘Minnesota’ ’ is used to talk about ‘Minnesota’. (6)

- Is ‘Minnesota’ used in (6)? What is mentioned in (6)?

One can use quotation marks to mention longer linguistic expressions or entire sentences. E.g.:

- ‘Snow is white’ is true just in case snow is white.
- ‘ ‘Snow is white’ is true just in case snow is white’ is true.

- Other ways of mentioning linguistic expressions:

  - By *displaying*:

Minnesota

has nine letters.

Tully was a Roman

is trochaic.

  - By *italicizing* (or otherwise highlighting):

- *Minnesota* has nine letters.
- Tully was a Roman is trochaic.

  - By *naming* (we can name anything we want, including linguistic expressions):

Let’s name ‘Tully was a Roman’ ‘Minnesota’. Minnesota has four words.

- Does ‘Minnesota’ have four words?
- Is Minnesota trochaic?
- Is ‘Minnesota’ trochaic?

Which of the following are true and which are false?

- Copper is copper.
- ‘Copper’ is the name of copper.
- The chemical symbol ‘Cu’ names ‘copper’.
- ‘Copper’ is copper.
- Copper is the name of copper.
- Some coins are made of copper.
- ‘Copper’ is a metal.

Object Language and Metalanguage

- For the most part, people use language to talk about things *other than* language.
- But in philosophy of language (also in logic, computer science, linguistics…), we often use language to talk *about language*.
- So we need to distinguish between the language we talk *about* (*object language*) and the language *in which* we talk about the object language (*metalanguage*).
- In a (metaphoric) sense, a metalanguage is "outside" or "external to" the object language.
- We can use English as a metalanguage to talk about German, French, Latin, Swahili, C++, or predicate logic. E.g.:

    'La neige est blanche' is a French sentence.

    '(∀x) (Fx ⊃ Gx' is not a syntactically correct expression in predicate logic: it lacks a parenthesis at the end.

And of course, we can use French as a metalanguage to talk about English (and the rest).

✓ But we can also use English as a metalanguage to talk about English as an object language. Witness:

'Snow is white' is true iff snow is white.

Compare it with:

'La neige est blanche' is true iff snow is white.

✓ The distinction between metalanguage and object language parallels the distinction between use and mention.

✓ When we use a metalanguage to study and talk about an object language, we *mention* expressions of the object language in sentences of our metalanguage.

In each of the following sentences 'Deutschland' is either used or mentioned. Indicate where that word is being used or mentioned.

- The only German word mentioned in the instructions to these exercises contains eleven letters.
- Some people think Deutschland and Germany are two different countries, but actually 'Deutschland' is the German name of Germany.
- 'Deutschland' is 'Deutschland'.
- The word 'Deutschland' is not being used in this sentence.
- Deutschland is the German name of Germany.

Consider this sentence (Cartwright 1984):

Some English words are obscene.

Are the following true or false?

- At least one word in the above sentence is obscene.
- At least one word in that sentence is 'obscene'.

Is 'Jack the Ripper' used or mentioned in the sentence below?

Jack the Ripper was so called because of his (rather regrettable) behavior.

Consider again:

Minnesota was the thirty-second state admitted to the Union. (1)
'Minnesota' has nine letters. (2)

'Minnesota' is used in (1) and mentioned in (2). (3)

- Is 'Minnesota' used or mentioned in (3)?

Which of the following are true and which are false?

- 'Minnesota' is part of the language.
- Minnesota is part of the language.
- ' 'Minnesota' has nine letters' is a sentence in a metalanguage.
- ' ' 'Minnesota' has nine letters' is a sentence in a metalanguage' is a sentence in a metalanguage.
- 'Minnesota' has nine letters is true.
- ' 'Minnesota' has nine letters' is true.
- ' 'Minnesota' has nine letters' is a sentence in a metalanguage.
- ' ' 'Minnesota' has nine letters' is true' is a sentence in a metalanguage.

Consider the sentence in the box:

La neige est blanche.

Which of the following are true and which are false?

- La neige is used in the sentence in the box.
- La neige is mentioned in the sentence in the box.
- ‘La neige’ is used in the sentence in the box.
- ‘La neige’ is mentioned in the sentence in the box.
- ‘La neige est blanche’ is the sentence in the box.
- The sentence in the box is a sentence in a metalanguage.
- The sentence in the box is false.
- ‘The sentence in the box is false’ is a sentence in a metalanguage.
- The sentence in the box means that snow is white.
- The sentence in the box means that la neige est blanche.
- The sentence in the box is la neige est blanche.
- The sentence in the box is *La neige est blanche*.
- The sentence in the box is

La neige est blanche.

Consider the sentence in the box:

The sentence in the box is false.

Which of the following are true and which are false?

- The sentence in the box means that the sentence in the box is false.
- The sentence in the box is true if and only if the sentence in the box is false.
- ‘The sentence in the box is false’ is true if and only if [=iff] the sentence in the box is false.
- ‘The sentence in the box is false’ is true iff the sentence in the box is true.
- The sentence in the box is mentioned in the previous two sentences.
- The sentence in the box is used in those two sentences.
- The sentence in the box is a sentence in the object language.
- The sentence in the box is a sentence in a metalanguage.

Metavariables and Corner Quotes

- A *metavariable* (usually a Greek letter) is an expression in the metalanguage that is used to talk *generally* about expressions of the object language – e.g., expressions of a certain *kind*.

Example: Propositional Logic. To explain the rule *Modus Ponens*, we need a way of talking about conditional sentences *generally* – a way of saying that whenever we have *any* sentence:

$$\Phi$$

and a conditional sentence of the *form*

$$\Phi \supset \Psi$$

we can introduce, as the next line of our derivation:

$$\Psi \qquad \text{(MP)}$$

Metavariables 'Φ' and 'Ψ' *stand for* any *arbitrary* sentences of Propositional Logic.

✓ Problem:

- Expressions such as '$\Phi \supset \Psi$' are *hybrid* expressions. Read properly, '$\Phi \supset \Psi$' *mentions* everything inside the quotation marks; i.e., it mentions both the logical symbol '$\supset$', which is part of our *language* and the Greek letters 'Φ' and 'Ψ', which do *not* belong to the language of Propositional Logic; strictly speaking, they are symbols (variables) of the *metalanguage* used to talk about PL. 'Φ' and 'Ψ' denote themselves inside quotation marks. This is *not* what we want! We want them to designate, not themselves, but *arbitrary sentences* of PL.
- We need a device similar to quotation marks; but such that it would treat symbols of the language and the metavariables *differently*: it should *mention* all the symbols of a *language* (e.g. PL) – let them function as their own names – but let the *metavariables* function as *names of arbitrary expressions* (of a certain kind, perhaps) in our *language*.

- Here it comes: ⌜ … ⌝ – “corner quotes,” “square quotes,” “quasi-quotes,” “Quine quotes.”

✓ Each symbol that occurs inside corner quotes is being used to designate some expression of a *language*.

- We only need square quotes when we have a *mixture* of symbols of a language and metavariables.

Modus Ponens: If Φ and ⌜Φ ⊃ Ψ⌝ are sentences of PL that occur earlier in a proof, one is allowed to write Ψ as the next line of the proof.

(Note that this rule is *purely syntactic*.)

Example: English. Suppose we want to talk generally about all English sentences beginning with ‘We’ and ending with ‘disaster’. The way to do it is to write:

⌜We Φ disaster⌝

where ‘Φ’ is any sequence of words within a grammatical English sentence.

Sentences and Propositions

- Earlier we said that the main semantic value of a *declarative sentence* is its truth or falsity. Now we need to qualify it.
- Many philosophers doubt that sentences *themselves* (i.e., linguistic entities: marks on paper, noises, etc.) are the sort of things that can be true or false. What is true or false is rather what a sentence *expresses*.

- What does a sentence express? – A *proposition*.

➢ Why do we need this distinction?

- The relation between sentences and what they express is *not* one-one.

  - Several different sentences may express the same proposition:
    - ‘Snow is white’
    - ‘La neige est blanche’
    - ‘Der schnee ist weiß’

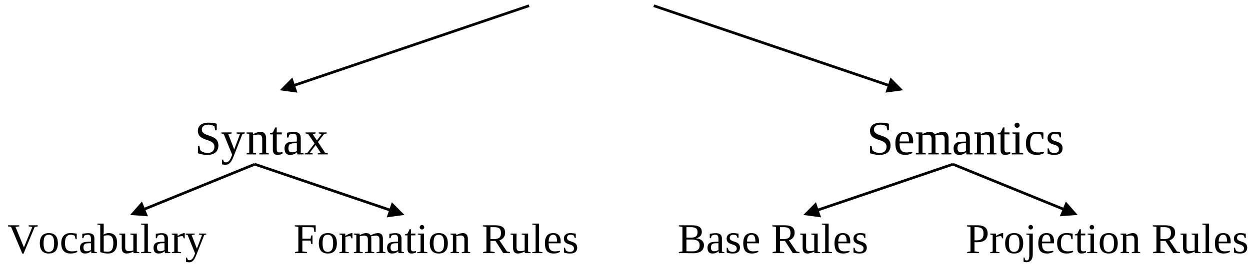
      – all express the proposition *that snow is white*.

  - The same sentence may express different propositions (and hence have different truth conditions) on different occasions of use:
    - ‘It is now noon’
    - ‘This is a fine red one’

  - Clearly, such sentences are neither true nor false *by themselves*.

➢ What is a proposition?

Formal Grammar for a Language



- Vocabulary: Specifies which marks or sounds can appear in sentences (or *be* sentences, in some cases).

- Formation Rules: Specify how sentences can be generated out of vocabulary items or out of shorter sentences.

- Formation Rules are in general *recursive*: they describe how *all and only* sentences (generally, an infinite number) of a given language can be generated by repeated application of a *finite* number of rules.

A Syntax for Languish (see Martinich)

- Vocabulary:
  - Proper names: 'Adam', 'Beth', 'Carol', 'David'
  - Predicates: 'walks', 'talks', 'flies', 'sits', 'reads'
  - Sentential connectives: 'it is not the case that', 'and', 'or', 'if … then', 'if and only if'
  - Punctuation marks: '(', ')'

- Formation Rules:
  1. Where α is any proper name and Φ is any predicate, enter $\alpha\Phi$ as a sentence. (Syntactic *base* rule)
  2. Where Φ is any sentence, enter $\ulcorner$it is not the case that $\Phi\urcorner$ as a sentence.
  3. Where Φ and Ψ are any sentences, enter $\ulcorner$(Φ and Ψ)$\urcorner$ as a sentence.
  4. Where Φ and Ψ are any sentences, enter $\ulcorner$(Φ or Ψ)$\urcorner$ as a sentence.
  5. Where Φ and Ψ are any sentences, enter $\ulcorner$(if Φ then Ψ)$\urcorner$ as a sentence.
  6. Where Φ and Ψ are any sentences, enter $\ulcorner$(Φ if and only if Ψ)$\urcorner$ as a sentence. (2–6 are syntactic *projection* rules)

➢ Any string of marks that can be generated by repeated application of Formation Rules is a sentence of Languish. No other string is a sentence of Languish. ("That's all, folks!")

Sentence generation in Languish

1.	Carol flies	Rule 1
2.	David sits	Rule 1
3.	(Carol flies or David sits)	Rule 4, from lines 1 and 2
4.	it is not the case that (Carol flies or David sits)	Rule 2, from line 3

✓ Remember: What is thus generated is a legitimate *string of marks* (or sounds) *devoid of any meaning*.

L1: The language consisting of all and only sequences of 'a's followed by the equal number of 'b's and vice versa.

Syntax for L1

- Vocabulary: a, b
- Formation Rules:
  1. Enter 'ab' as a sentence.
  2. Enter 'ba' as a sentence. (1–2 are syntactic base rules)
  3. If ⌈aΦb⌉ is a sentence, enter ⌈aaΦbb⌉ as a sentence.
  4. If ⌈bΦa⌉ is a sentence, enter ⌈bbΦaa⌉ as a sentence. (3–4 are syntactic *projection* rules)

L2: The language consisting of sequences of 'a's and 'b's followed by the mirror image of 'a's and 'b's. (E.g.: 'baab', 'abaaba', 'bbbb'; but not 'abab', 'bab', 'a'

Syntax for L2

- Vocabulary: a, b
- Formation Rules:
  1. Enter 'aa' as a sentence.
  2. Enter 'bb' as a sentence.
  3. If Φ is a sentence, enter ⌈aΦa⌉ as a sentence.
  4. If Φ is a sentence, enter ⌈bΦb⌉ as a sentence.

Generation of 'abbaabaabaabba'

1. aa	Rule 1
2. baab	Rule 4, from line 1
3. abaaba	Rule 3, from line 2
4. aabaabaa	Rule 3, from line 3
5. baabaabaab	Rule 4, from line 4
6. bbaabaabaabb	Rule 4, from line 5
7. abbaabaabaabba	Rule 3, from line 6

L3: The language of the numerals for natural numbers: 15, 2000506 …

- Vocabulary: 1, 2, 3, 4, 5, 6, 7, 8, 9, 0
- Formation Rules:
  1. Enter ‘1’ as a sentence.
  2. Enter ‘2’ as a sentence.
  3. Enter ‘3’ as a sentence.
  4. Enter ‘4’ as a sentence.
  5. Enter ‘5’ as a sentence.
  6. Enter ‘6’ as a sentence.
  7. Enter ‘7’ as a sentence.
  8. Enter ‘8’ as a sentence.
  9. Enter ‘9’ as a sentence.
  10. Where Φ is a sentence and Ψ is a vocabulary item, enter ΦΨ as a sentence.

  ➢ *Numerals* versus *Numbers*

From Syntax to Semantics

- How to impart meaning to our strings of marks or sounds?
  - By connecting them to the world!

- Specify the meanings of the basic elements of a language (either words or simplest sentences).
- Introduce the rules stipulating how the meaning of more complex expressions is determined by the meaning of its constituents.

- Example: Truth-table technique in Sentential (Propositional) Logic (SL).
  - Assign truth-values to the atomic sentences of SL.
  - Apply characteristic truth-tables for sentential connectives, to determine the truth-value of a complex sentence.

E	F	J	(J	&	[(	E	∨	F)	&	(~	E	&	~	F)])	⊃	~	J)
T	T	T	T	F		T	T	T	F	F	T	F	F	T	**T**	F	T

Example: The Semantics of Sentential Logic

Def: A *truth-value assignment* (TVA) is a function $\mathcal{A}$ from the set of all sentence letters (atomic sentences) $\boldsymbol{\mathcal{S}}_{\text{atom}}$ of SL into the set {**T**, **F**}:

$$\mathcal{A}: \boldsymbol{\mathcal{S}}_{\text{atom}} \mapsto \{\mathbf{T}, \mathbf{F}\}$$

Def: *Truth and falsity on a TVA*

1. If $\mathbf{P} \in \boldsymbol{\mathcal{S}}_{\text{atom}}$ then **P** is true on $\mathcal{A}$ if $\mathcal{A}(\mathbf{P}) = \mathbf{T}$, and **P** is false on $\mathcal{A}$ if $\mathcal{A}(\mathbf{P}) = \mathbf{F}$.
2. **~P** is true on $\mathcal{A}$ if **P** is false on $\mathcal{A}$; **~P** is false on $\mathcal{A}$ if **P** is true on $\mathcal{A}$;
3. (**P**&**Q**) is true on $\mathcal{A}$ if **P** is true on $\mathcal{A}$ and **Q** is true on $\mathcal{A}$; otherwise (**P**&**Q**) is false on $\mathcal{A}$.
4. (**P**∨**Q**) is true on $\mathcal{A}$ if **P** is true on $\mathcal{A}$ or **Q** is true on $\mathcal{A}$; otherwise (**P**∨**Q**) is false on $\mathcal{A}$.
5. (**P**⊃**Q**) is true on $\mathcal{A}$ if **P** is false on $\mathcal{A}$ or **Q** is true on $\mathcal{A}$; otherwise (**P**⊃**Q**) is false on $\mathcal{A}$.
6. (**P**≡**Q**) is true on $\mathcal{A}$ if **P** and **Q** are both true on $\mathcal{A}$, or **P** and **Q** are both false on $\mathcal{A}$; otherwise (**P**≡**Q**) is false on $\mathcal{A}$.

Def: A sentence **P** of SL is *truth-functionally true* ($\vDash_{SL}$**P**, or simply $\vDash$**P** where only SL is under discussion) iff **P** is true on every TVA.

Def: A sentence **P** of SL is *truth-functionally false* (**P**$\vDash$) iff **P** is false on every TVA.

Def: A sentence **P** of SL is *truth-functionally indeterminate* iff it is neither truth-functionally true nor truth-functionally false.

Def: Sentences **P** and **Q** of SL are *truth-functionally equivalent* (**P** $\Dashv\vDash$ **Q**) iff **P** and **Q** have the same truth value on every TVA.

Def: A set **Γ** of sentences of SL is *truth-functionally consistent* (**Γ**$\nvDash$) iff there is a TVA on which every member of **Γ** is true. A set **Γ** of sentences of SL is *truth-functionally inconsistent* (**Γ**$\vDash$) iff it is not truth-functionally consistent.

Def: A set **Γ** of sentences of SL *truth-functionally entails* a sentence **P** (**Γ** $\vDash$ **P**) iff there is no TVA on which every member of **Γ** is true and **P** is false.

Def: An argument of SL is *truth-functionally valid* iff there is no TVA on which all its premises are true and the conclusion is false.

Corollary: An argument of SL is truth-functionally valid iff the set consisting of its premises truth-functionally entails its conclusion.

A Semantics for the Language of Numerals for Natural Numbers

- Semantic Base Rules:
  1. ‘1’ refers to the number one.
  2. ‘2’ refers to the number two.

     …

  9. ‘9’ refers to the number nine.
  10. ‘0’ refers to the number zero.

- Semantic Projection Rule:
  11. Where ΦΨ is a sentence, Φ is a string of vocabulary items, and Ψ is a vocabulary item, ΦΨ refers to ten times the number referred to by Φ, plus the number referred to by Ψ.

	Sentence	Semantics	Semantic Rule
1.	7	7	Rule 7
2.	79	$10 \times 7 + 9$	Rule 11, line 1
3.	796	$10 \times (10 \times 7 + 9) + 6$	Rule 11, line 2
4.	7964	$10 \times ((10 \times 7 + 9) + 6) + 4$	Rule 11, line 3

- “Sentences” of this language are very unlike the sentences of English or even of logical languages; they are singular terms that have reference (refer to natural numbers) but not truth value: they do not express any propositions.
- A semantics for a more “natural” language should explain how the reference of its singular terms and of its predicate expressions come together to *determine* the truth value of declarative sentences they generate.
- But the general idea of *compositionality* remains the same: the semantic value of the whole (e.g., the truth value of a sentence) is determined by the semantic values of its parts (e.g., the reference of its singular terms and predicates).

A Semantics for Languish

Semantic:	Proper Names:	Predicates:
Base	‘Peter’ refers to Peter	‘is articulate’ refers to the set {Peter, Sue}
Rules	‘Richard’ refers to Richard	‘chuckles’ refers to the set {Richard}
	‘Sue’ refers to Sue	‘stumbles’ refers to the set {Richard}

Semantic Projection Rules:

1. Where α is a proper name and Φ is a predicate, the sentence $\alpha\Phi$ is true if and only if the object referred to by α is a member of the set referred to by Φ.
2. The sentence $\ulcorner$it is not the case that $\Phi\urcorner$ is true iff Φ is false; otherwise it is false.
3. The sentence $\ulcorner(\Phi$ and $\Psi)\urcorner$ is true iff both Φ and Ψ are true; otherwise it is false.
4. The sentence $\ulcorner(\Phi$ or $\Psi)\urcorner$ is false iff both Φ and Ψ are false; otherwise it is true.
5. The sentence $\ulcorner$(if Φ then $\Psi)\urcorner$ is false iff Φ is true and Ψ is false; otherwise it is true.
6. The sentence $\ulcorner(\Phi$ if and only if $\Psi)\urcorner$ is true iff both Φ and Ψ are true or both are false; otherwise it is false.

(Richard chuckles and Sue stumbles) if and only if it is not the case that Peter is articulate

Sentence	Syntax	Semantics
1. Richard chuckles	Rule 1	True by Rule 1
2. Sue stumbles	Rule 1	False by Rule 1
3. (Richard chuckles and Sue stumbles)	Rule 3	False by Rule 3
4. Peter is articulate	Rule 1	True by Rule 1
5. it is not the case that Peter is articulate	Rule 2	False by Rule 2
6. (Richard chuckles and Sue stumbles) if and only if it is not the case that Peter is articulate	Rule 6 from lines 3 and 5	True by Rule 6 from lines 3 and 5

- The expressive power of Languish is limited: lacks *quantifiers* and *many-place* predicates…

First-Order Logic (Predicate Logic with Identity)

Quantifiers:

Universal Quantifier: (∀x) Hx

- For all x, x is happy
- Everyone (or everything in the Universe of Discourse) is happy

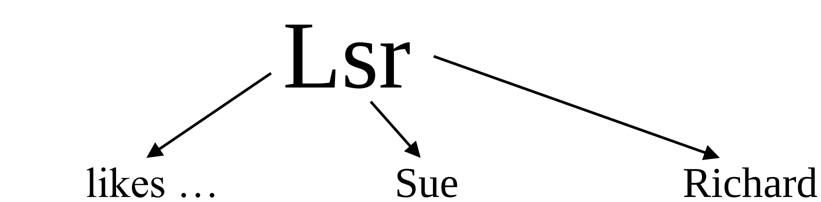
Existential Quantifier: (∃x) Hx

- There is an a such that x is happy
- Someone (or something in the Universe of Discourse) is happy

Many-place predicates refer to *relations* (just as one-place predicates refer to properties).

Universe of Discourse (UD): {Peter, Richard, Sue, Tom, Cynthia}

- Lsr: Sue likes Richard



➢ Order matters!

(∀x) Lxs: Everyone (everything in UD) likes Sue
(∀x) Lsx: Sue likes everyone (everything in UD)
(∃x) Lxc: Someone likes Cynthia
~ (∃x) Lcx: Cynthia doesn't like anyone.
(∀x) (∃y) Lxy: Everyone likes someone.
(∃x) (∀y) Lxy: Someone likes everyone.
(∀x)Lcx ⊃ ((∀x) Hxr ∨ (∃x) Lxp): If Cynthia likes everyone, then everyone hates Richard or someone likes Peter.

- Two-place predicates refer to two-place *relations*. Similarly for *n*-place predicates.

- $\Phi\alpha\beta$ is true iff the ordered pair of objects referred to by α and β is a member of the set referred to by Φ.
- $\ulcorner(\forall\upsilon)\Phi\upsilon\urcorner$ is true iff $\Phi\alpha$ is true for every assignment of an object in UD to α.
- $\ulcorner(\exists\upsilon)\Phi\upsilon\urcorner$ is true iff $\Phi\alpha$ is true for at least one assignment of an object in UD to α.

An Interpretation for FOL consists in:

- the specification of a UD (a non-empty set of objects);
- the assignment of a truth-value to each sentence letter (a zero-place predicate);
- the assignment of a member of the UD to each individual constant (proper name);
- the assignment of a set of ordered *n*-tuples of members of the UD (extensions) to each *n*-place predicate.

A Model for a sentence is an Interpretation that makes the sentence true.

Example: Formal Semantics of FOL

Def: An *n*-place *function* $\mathfrak{F}$ *on a universe of discourse* $\mathcal{U}_{\mathbf{I}}$ of an interpretation **I** is a function that maps every *n*-tuple of members of $\mathcal{U}_{\mathbf{I}}$ to a single member of $\mathcal{U}_{\mathbf{I}}$:

$$\mathfrak{F}: \quad \mathcal{U}_{\mathbf{I}}^{n} \mapsto \mathcal{U}_{\mathbf{I}},$$

where $\mathcal{U}_{\mathbf{I}}^{n} =_{\text{def}} \mathcal{U}_{\mathbf{I}} \times \mathcal{U}_{\mathbf{I}} \times \ldots \times \mathcal{U}_{\mathbf{I}}$ (*n* times)

Def: An *interpretation* **I** of PLE is an ordered pair $\langle \mathcal{U}, \boldsymbol{v} \rangle$, where $\mathcal{U}$ is a non-empty set and $\boldsymbol{v}$ is a function that meets the following conditions:

1. $\boldsymbol{v}(\mathbf{P}) \in \{\mathbf{T}, \mathbf{F}\}$ for every sentence letter **P** of PLE;
2. $\boldsymbol{v}(\mathbf{a}) \in \mathcal{U}$ for every individual constant **a** of PLE;
3. $\boldsymbol{v}(\boldsymbol{f}) = \mathfrak{F} \subseteq \mathcal{U}^{n} \times \mathcal{U}$ for every functor $\boldsymbol{f}$ of PLE, where $\mathfrak{F}$ is an *n*-place function on $\mathcal{U}$;
4. $\boldsymbol{v}(\mathbf{A}) \subseteq \mathcal{U}^{n}$ for every *n*-place predicate (predicate letter) **A** of PLE;

If **I** is an interpretation of PL and $\mathbf{I} = \langle \mathcal{U}, \boldsymbol{v} \rangle$, then $\mathcal{U}_{\mathbf{I}} = \mathcal{U}$ and $\boldsymbol{v}_{\mathbf{I}} = \boldsymbol{v}$.

Def: A *variable assignment* **d** (an *assignment* of values to variables) for an interpretation **I** is a function that assigns to each variable **x** of PLE a member **d**(**x**) of $\mathcal{U}_{\mathbf{I}}$.

Def: A *variant* **d**[**u/x**] of an assignment **d** for an interpretation **I**, where $\mathbf{u} \in \mathcal{U}_{\mathbf{I}}$, is defined as follows: for every variable **x′**,

if **x′** = **x**, then **d**[**u/x**](**x′**) = **u**;

if **x′** ≠ **x**, then **d**[**u/x**](**x′**) = **d**(**x′**).

(By convention, $\mathbf{d}[\mathbf{u}_1/\mathbf{x}_1, \ldots \mathbf{u}_n/\mathbf{x}_n] =_{\text{df}} \mathbf{d}[\mathbf{u}_1/\mathbf{x}_1] \ldots [\mathbf{u}_n/\mathbf{x}_n]$.)

Def: For any individual term **t** of PL and any interpretation **I** and any assignment **d** for **I**, define $\mathscr{r}(\mathbf{t}, \mathbf{d}, \mathbf{I})$ as follows:

if **t** is a variable, then $\mathscr{r}(\mathbf{t}, \mathbf{d}, \mathbf{I}) = \mathbf{d}(\mathbf{t})$;

if **t** is an individual constant, then $\mathscr{r}(\mathbf{t}, \mathbf{d}, \mathbf{I}) = \mathscr{v}_{\mathbf{I}}(\mathbf{t})$;

if **t** is an individual term $\boldsymbol{f}(\mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_n)$, where $\boldsymbol{f}$ is an *n*-place functor and $\mathbf{t}_1, \mathbf{t}_2, \ldots \mathbf{t}_n$ are individual terms of PLE, then if $\langle \mathscr{r}(\mathbf{t}_1, \mathbf{d}, \mathbf{I}), \ldots \mathscr{r}(\mathbf{t}_n, \mathbf{d}, \mathbf{I}), \mathbf{u} \rangle \in \mathscr{v}_{\mathbf{I}}(\boldsymbol{f})$, then $\mathscr{r}(\mathbf{t}, \mathbf{d}, \mathbf{I}) = \mathbf{u}$.

Def: *Satisfaction* (**d**,**I** ⊨ **P**, read "**d** satisfies **P** relative to **I**") is a (three-place) relation between an interpretation **I** of PL, an assignment **d** for that interpretation, and a formula **P** of PLE.

1. If **P** is a sentence letter of PLE, then **d**,**I** ⊨ **P** iff $\mathcal{v}_{\mathbf{I}}(\mathbf{P}) = \mathbf{T}$;
2. If **P** is an atomic formula of the form $\mathbf{A}\mathbf{t}_1\ldots\mathbf{t}_n$, then **d**,**I** ⊨ **P** iff $\langle \mathcal{r}(\mathbf{t}_1, \mathbf{d}, \mathbf{I}), \ldots \mathcal{r}(\mathbf{t}_n, \mathbf{d}, \mathbf{I})\rangle \in \mathcal{v}_{\mathbf{I}}(\mathbf{A})$.
3. If **P** is an atomic formula of the form $\mathbf{t}_1 = \mathbf{t}_2$, then **d**,**I** ⊨ **P** iff $\mathcal{r}(\mathbf{t}_1, \mathbf{d}, \mathbf{I}) = \mathcal{r}(\mathbf{t}_2, \mathbf{d}, \mathbf{I})$.
4. If **P** and **Q** are formulas of PLE, then:

   d,**I** ⊨ **~P** iff **d**,**I** ⊭ **P**;

   d,**I** ⊨ **P**&**Q** iff **d**,**I** ⊨ **P** and **d**,**I** ⊨ **Q**;

   d,**I** ⊨ **P**∨**Q** iff **d**,**I** ⊨ **P** or **d**,**I** ⊨ **Q**;

   d,**I** ⊨ **P**⊃**Q** iff **d**,**I** ⊭ **P** or **d**,**I** ⊨ **Q**;

   d,**I** ⊨ **P**≡**Q** iff either **d**,**I** ⊨ **P** and **d**,**I** ⊨ **Q** or **d**,**I** ⊭ **P** and **d**,**I** ⊭ **Q**.
5. **d**,**I** ⊨ (∀**x**)**Q** iff **d**[**u**/**x**],**I** ⊨ **Q** for every $\mathbf{u} \in \mathcal{U}_{\mathbf{I}}$;
6. **d**,**I** ⊨ (∃**x**)**Q** iff **d**[**u**/**x**],**I** ⊨ **Q** for some $\mathbf{u} \in \mathcal{U}_{\mathbf{I}}$.

Def: A sentence **P** is *true on an interpretation* **I** iff **d**,**I** $\vDash$ **P** for every assignment **d** defined for **I**.

Def: A sentence **P** is *false on an interpretation* **I** iff **d**,**I** $\nvDash$ **P** for every assignment **d** defined for **I**.

PL3: For any interpretation **I** and any sentence **P** of PLE, either every assignment for **I** satisfies **P** (**d**,**I** $\vDash$ **P**) or no assignment for **I** satisfies **P** (**d**,**I** $\nvDash$ **P**).

Corollary: If **P** is a sentence of PLE, then for every **I**, either **P** is true on **I** or **P** is false on **I** (but not both).

Def: A set of sentences **Γ** *quantificationally entails* a sentence **P** (**Γ** ⊨ **P**) iff **P** is true on every interpretation on which all members of **Γ** are true.

Def: A set of sentences **Γ** is *quantificationally consistent* iff there is an interpretation on which all members of **Γ** are true.

Def: A sentence **P** is *quantificationally true* (⊨**P**) iff **P** is true on every interpretation.

Def: A sentence **P** is *quantificationally false* (**P**⊨) iff **P** is false on every interpretation.

Def: Sentences **P** and **Q** are *quantificationally equivalent* iff for every interpretation **I**, **P** and **Q** are either both true or both false on **I**.

Example: I1: Interpretation of a fragment of FOL

UD: Henry, Michael, Rita, Sue
‘h’ refers to Henry
‘m’ refers to Michael
‘r’ refers to Rita
‘s’ refers to Sue
‘E’ (“easygoing”) refers to {Henry, Sue}
‘L’ (“likes”) refers to {<Michael, Michael>, <Michael, Rita>, <Michael, Sue>, <Sue, Henry>}

- Determine the truth-value of the symbolization of ‘Everyone whom Michael likes is easygoing’ in FOL on I1: (∀x) (Lmx ⊃ Ex) (tedious!)

- Check the truth-value of ‘(Lmx ⊃ Ex)’ for all assignments of objects in UD to ‘x’:
  - Assign Henry to ‘x’.
    - ‘Lmh’ is false because the ordered pair of objects referred to by ‘m’ and ‘h’ (i.e., <Michael, Henry>) is not a member of the set referred to by ‘L’ (i.e., {<Michael, Michael>, <Michael, Rita>, {<Michael, Sue>, <Sue, Henry>}).
    - ‘Eh’ is true because the object referred to by ‘h’ (i.e., Henry) is a member of the set referred to by ‘E’ (i.e., {Henry, Sue})
    - Since ‘Lmh’ is false and ‘Eh’ is true, ‘(Lmh ⊃ Eh)’ is true.
  - Assign Michael to ‘x’. … ‘(Lmm ⊃ Em)’ is false. (Strictly speaking, we can stop here (why?); but let’s continue building a character!)
  - Assign Rita to ‘x’. … ‘(Lmr ⊃ Er)’ is false.
  - Assign Sue to ‘x’. … ‘(Lms ⊃ Es)’ is true.

- It is not the case that ‘(Lmx ⊃ Ex)’ is true for all assignments of objects in UD to ‘x’. Therefore, ‘(∀x) (Lmx ⊃ Ex)’ is false. I1 is not a model for this sentence.

Identity

◈ ‘=’ a very mysterious equivalence relation!

Leibniz’s Principles:

- *Indiscernibility of Identicals*: If x is identical with y, then x and y have all the same properties: $(x = y) \rightarrow (\forall\Phi)\,(\Phi x \leftrightarrow \Phi y)$
- *Identity of Indiscernibles*: If a and y have all the same properties, then they are identical: $(\forall\Phi)\,(\Phi x \leftrightarrow \Phi y) \rightarrow (x = y)$
- Corollary: If there is a property that x has and y does not, then x is not identical with y: $(\exists\Phi)\,(\Phi x \,\&\, {\sim}\,\Phi y) \rightarrow (x \neq y)$

➢ Problems down the road:

Aspirin is known by John to be a pain reliever.
Acetylsalicylic acid is not known by John to be a pain reliever.
∴ Aspirin ≠ acetylsalicylic acid.

The Morning Star = The Morning Star

versus

The Morning Star = The Evening Star